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Title of Presentation:
Area Inside a Circle: Intuitive and Rigorous Proofs

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In the following we present a brief review of the proofs of area inside a circle. A typical rigorous proof requires knowlege of integral calculus. But even in these proofs presented by calculus books the authors resort to circular reasoning.

To prove the area inside a circle, they set up the integral $\int_0^1 \sqrt{1-x^2}dx$ followed by trigonometric substitution which requires knowing that the derivative of $\sin\theta$ is $\cos\theta$. But this latter fact requires proving that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$. For this proof, they resort to a geometric argument, bounding the area of a sector of a unit circle between the areas of two triangles and showing that $\sin\theta \le \theta \le \tan\theta$. They then apply the Squeeze Theorem. But for computation of the sector's area, they resort to a standard formula, $A = \frac{1}{2}\theta$, which is based on knowing the area of a circle. So, they prove the area by assuming the area. This is obviously circular argumentation!

In this short piece we begin by proving a preliminary result showing that $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$, without, a priori, assuming the area of a sector. This limit is central to the proof of the derivatives of trigonometric functions. We note that aside from the aforementioned limit, the function $\dfrac{\sin\theta}{\theta}$ itself plays an important role not only in mathematics but in other fields of science such as physics and engineering.

Consider a circle of radius 1, centered at the origin, as shown in Fig. 1.

Theorem 1. Let $0 < \theta < \frac{\pi}{2}$ be an angle measured in radians. Then, $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

Proof. Since the magnitude of θ equals the length of the arc it subtends and since $\sin\theta < \overline{AD}$, we have that $\sin\theta < \theta$, or $1 < \dfrac{\theta}{\sin\theta}$. This establishes a lower bound for $\dfrac{\theta}{\sin\theta}$. To show the upper bound, observe that by the triangle inequality, $\theta < \sin\theta + \overline{BD}$. This can be established by the standard method of estimating an arc length of a rectifiable curve by the linear approximation of the lengths of the chords it subtends through partitioning. The result follows by applying the triangle inequality in each partition.

Noting that $\sin\theta < \tan\theta$, we get $\theta < \tan\theta + \overline{BD}$. But $\overline{BD} = 1 - \cos\theta$. So, $\dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta} + \dfrac{1-\cos\theta}{\sin\theta}$ and $\dfrac{1-\cos\theta}{\sin\theta} = \sqrt{\dfrac{(1-\cos\theta)^2}{1-\cos^2\theta}} = \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$. Combining this result with the previous lower bound gives,
$1 < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta} + \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$. Letting $\theta \to 0$ in the last expression completes the upper bound, resulting in $1 \leq \dfrac{\theta}{\sin\theta} \leq 1$.
Finally, applying the Squeeze Theorem we get,

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

$(*)$

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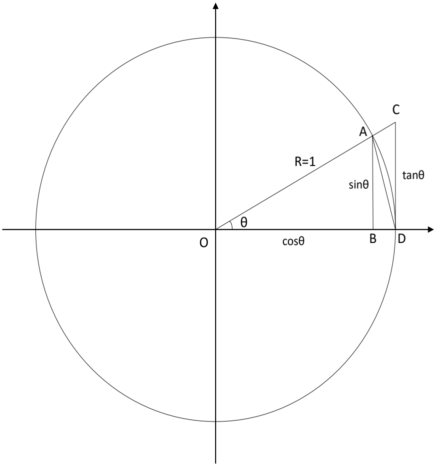

Fig. 1

An interesting question related to the foregoing bounding of the angle θ is that if we define the derivatives of trigonometric functions of θ analytically (i.e., by infinite series or complex numbers or solutions of differential equations), can we arrive at the bounding of the angle? The next theorem follows.

Theorem 2. If $(\sin\theta)' = \cos\theta$, and $(\tan\theta)' = \sec^2\theta$, then $\sin\theta \leq \theta \leq \tan\theta$.

Proof. Let $f(\theta) = \sin\theta - \theta$.

$f(0) = 0$ and $f^{'}(\theta) = \cos\theta - 1 \leq 0$, giving $f^{'}(\theta) < 0$ for $\theta > 0$.

Hence,

$$f(\theta) = \int_0^{\theta} f^{'}(t)dt < 0.$$

Therefore, $\sin\theta < \theta$. This gives a lower bound for θ.

To find an upper bound for θ, let $g(\theta) = \tan\theta - \theta$.

$g(0) = 0$ and $g^{'}(\theta) = \sec^2\theta - 1 \geq 0$, giving $g^{'}(\theta) > 0$ for $\theta > 0$.

Hence,

$$g(\theta) = \int_0^\theta g^{'}(t)dt > 0.$$

Therefore, $\theta < \tan\theta$. This gives an upper bound for θ. Combining the

above results we get $\sin\theta \leq \theta \leq \tan\theta$.

Theorem 3. Area inside a circle of radius R is πR^2.

Proof. Consider a circle of radius R centered at the origin in Fig. 2. Partition the circle into n equal slices and consider a slice with central angle $\frac{2\pi}{n}$ radians. We know that the area of a triangle is one-half times the product of two of its sides times the sine of the angle between the two sides. So, the area of the triangle subtended by the central angle $\frac{2\pi}{n}$ becomes $A = \frac{1}{2}R^2 \sin(\frac{2\pi}{n})$. Because there are n inscribed triangles in the circle, the total area of all these triangles would be $A_{\text{total}} = n\frac{1}{2}R^2 \sin(\frac{2\pi}{n})$. As we increase the number of slices by increasing n, the sum of the areas of the inscribed triangles get closer to the area of the circle. To get the area of the circle, we need to find the limit of A_{total}, as $n \to \infty$. So, using $(*)$, and since $\frac{2\pi}{n} \to 0$, as $n \to \infty$, the area of the circle becomes: $A_{\text{circle}} = \lim_{n\to\infty}$

$$n\frac{1}{2}R^2 \sin(\frac{2\pi}{n}) = \pi R^2 (\lim_{n\to\infty} \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}}) = \pi R^2.$$

Hence, $A_{\text{circle}} = \pi R^2$.

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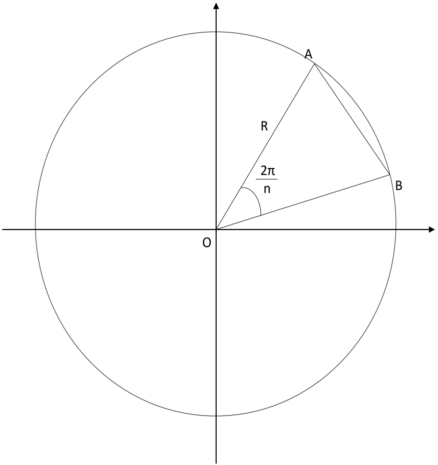

Fig. 2

There are many other intuitive approaches also, some of which involve slicing or opening up a circle. Below we offer a simple intuitive proof which is not based on area stretching, but assumes area preservation under mappings. Consider two concentric circles with radii r and R and corresponding areas A_r and A_R. Cut the annulus open in the shape of a right angle trapezoid $ABCD$ as in Fig. 3.

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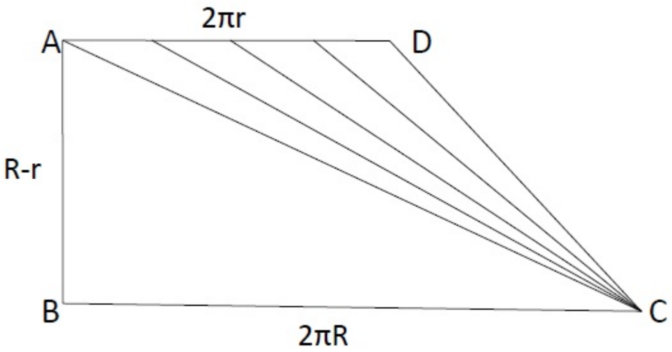

Fig. 3

We can see that the area of the annulus equals the area of the trapezoid. So, $A_{\text{annulus}} = A_{\text{trapezoid}}$, or $A_R - A_r = \frac{1}{2}(2\pi R + 2\pi r)(R - r) = \pi R^2 - \pi r^2$. We can choose $r > 0$ as small as we please and so, in particular, if we let r approach 0, the area of the inner circle approaches 0 and we get, $A_R - 0 = \pi R^2 - 0$. Thus $A_R = \pi R^2$.
Note that shrinking r to 0, shrinks the trapezoid to the right triangle ABC, whose area is $\frac{1}{2}(2\pi R)(R - 0) = \pi R^2 = A_R$.
In the following we present an analytic proof of the area inside a circle using area stretching, which does not assume area preserving mapping of regions.

Theorem 4. Area inside a circle of radius r is πr^2.

Proof. Consider a circle of radius r centered at the origin and partition it into n equal sectors, each having central angle $\dfrac{2\pi}{n}$, and the coresponding arc length $\dfrac{2\pi}{n}r$. Assume the area of a sector is c_n. If we stretch the radius r by a factor of $k > 1$, we create a circle with radius $R = kr$. So, the corresponding streched sector will have an arc length equal to $\dfrac{2\pi}{n}kr$ and the its area will be increased by a factor of k^2 to $k^2 c_n$; see Fig. 4 Now the area between the two sectors is $A_{\text{between sectors}} = k^2 c_n - c_n = c_n[k^2 - 1]$ which is approximately equal to the area of the trapezoid $ABCD$, in Fig. 4.

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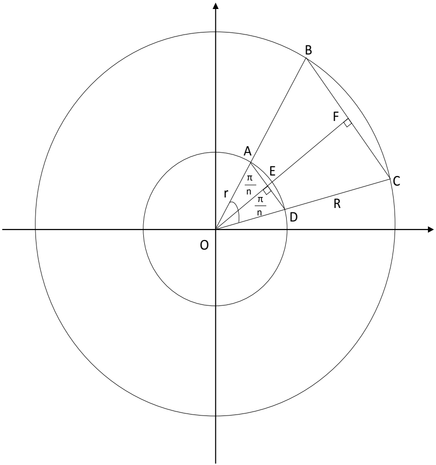

Fig. 4

If we connect the center O to the point F which is the midpoint of $\overline{BC}$, the triangles ΔOBF and ΔOCF become right angle congruent triangles with right angles at the point F. As a result, central angles $\angle AOE$ and $\angle DOE$ will each equal $\frac{\pi}{n}$. Obviously triangles ΔOAE and ΔODE are also congruent having right angles at the point E. To calculate the area of the trapezoid, we note that its larger base has length $\overline{BC} = 2R\sin\frac{\pi}{n}$, and its smaller base has length $\overline{AD} = 2r\sin\frac{\pi}{n}$. The height of the trapezoid is $\overline{EF} = R\cos\frac{\pi}{n} - r\cos\frac{\pi}{n}$. Therefore the area of the trapezoid becomes:

$$\begin{aligned} A_{\text{trapezoid}} &= \tfrac{1}{2}[2R\sin\frac{\pi}{n} + 2r\sin\frac{\pi}{n}]\cdot[R\cos\frac{\pi}{n} - r\cos\frac{\pi}{n}] \\ &= \sin\frac{\pi}{n}\cos\frac{\pi}{n}[R^2 - r^2] \\ &= \sin\frac{\pi}{n}\cos\frac{\pi}{n}\cdot r^2[k^2 - 1]. \end{aligned}$$

Setting $A_{\text{between sectors}} \approx A_{\text{trapezoid}}$, gives,
$c_n[k^2 - 1] \approx \sin\frac{\pi}{n}\cos\frac{\pi}{n}\cdot r^2[k^2 - 1]$, or $c_n \approx \sin\frac{\pi}{n}\cos\frac{\pi}{n}\cdot r^2$.
This approximation can be improved by increasing n. Now, multiplying both sides of the above by n gives,

$$nc_n \approx n\sin\frac{\pi}{n}\cos\frac{\pi}{n}\cdot r^2.$$

Since there are exactly n identical sectors in the circle of radius r, its area becomes $c = nc_n$.

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Therefore,
$c \approx n \sin \dfrac{\pi}{n} \cos \dfrac{\pi}{n} \cdot r^2.$

Now, taking the limit of both sides as $n \to \infty$, and applying our earlier result $(*)$ and the fact that $\cos\theta \to 1$, as $\theta \to 0$, we get

$$c = \lim_{n\to\infty} n \sin \frac{\pi}{n} \cos \frac{\pi}{n} \cdot r^2 = \lim_{n\to\infty} \pi \cdot \frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}} \cos \frac{\pi}{n} \cdot r^2$$

$$= \pi r^2 (\lim_{n\to\infty} \frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}}) \cdot (\lim_{n\to\infty} \cos \frac{\pi}{n}) = \pi r^2.$$

Hence, $c = \pi r^2$.